International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 3, Issue 2, Mar 2014, 21-28

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

COMMON FIXED POINT THEOREM FOR FINITE NUMBER OF WEAKLY COMPATIBLE MAPPINGS IN QUASI-GAUGE SPACE

V. K. GUPTA & GARIMA SAXENA

Department of Mathematics, Govt. Madhav Science College, Ujjain, Madhaya Pradesh, India

ABSTRACT

The present paper deal with common fixed point theorems for finite number of weak compatible mapping in Quasi- gauge space. By pointing out the fact that the continuity of any mapping for the existence of the fixed point is not necessary, we improve the result of Rao and Murthy [6].

KEYWORDS: Common Fixed Point, Quasi-Gauge Space, Weakly Compatible Mappings

1. INTRODUCTION

The concept of quasi-gauge space is due to Reilly [7] in the year 1973. Afterwards, Antony et. al. [2] gave a generalization of a common fixed point theorem of Fisher [3] for quasi-gauge spaces. Pathak et. al. [5] proved fixed point theorems for compatible mappings of type (P). Rao and Murthy [6] extended results on common fixed points of self maps by replacing the domain “complete metric space” with “Quasi-gauge space”. But in both theorems continuity of any mapping was the necessary condition for the existence of the fixed point. We improve results of Rao and Murthy [6] and show that the continuity of any mapping for the existence of the fixed point is not required.

Definition 1.1: A Quasi-pseudo-metric on a set X is a non negative real valued function p on $X \times X$ such that

- $p(x,x) = 0$ for all $x \in X$.
- $p(x,z) \le p(x,y) + p(y,z)$ for all $x,y,z \in X$.

Definition 1.2: A Quasi-gauge structure for a topological space (X, T) is a family P of quasi-pseudo-metrics on X such that T has as a sub-base the family

$\{B(x, p, \varepsilon) : x \in X, p \in P, \varepsilon > 0\}$

Where $B(x, p, \varepsilon)$ is the set $\{y \in X: p(x, y) < \varepsilon\}$. If a topological space has a Quasi-gauge structure, it is called a quasi-gauge space.

Definition 1.3: A sequence $\{x_n\}$ in a Quasi-gauge space (X, P) is said to be P-Cauchy, if for each $\varepsilon>0$ and $p\in P$ there is an integer k such that $p(x_m, x_n) < \varepsilon$ for all $m, n \ge k$.

Definition 1.4: A Quasi-gauge space (X, P) is sequentially complete iff every P-Cauchy sequence in X is convergent in X.

We now propose the following characterization. Let (X, P) be a Quasi-gauge space X is a T_0 Space iff $p(x,y) = p(x, y) = 0$ for all p in P implies $x = y$.

Antony [1] introduced the concept of weak compatibility for a pair of mappings on Quasi-gauge Space.

Definition 1.5: Let (X, P) be a Quasi-gauge space. The self maps f and g are said to be (f, g) weak compatible if lim $gfx_n = fz$ for some $z \in X$ whenever x_n is sequence in X such that

$\lim fx_n = \lim gx_n = z$ and $\lim fgx_n = \lim ffx_n = fz$.

f and g are said to be weak compatible to each other if (f, g) and (g, f) are weak compatible.

Lemma 1.1: [4] Suppose that ψ: $[0, \infty) \to [0, \infty)$ is non-decreasing and upper semi-continuous from the right. If $\psi(t) < t$ for every $t > 0$, then $\lim \psi^n(t) = 0$.

Rao and Murty [6] proved the following.

Theorem 2: Let A, B, S and T be self maps on a left (right) sequentially complete Quasi-gauge T_0 space (X,P) such that

- (A,S), (B,T) are weakly compatible pairs of maps with $T(X) \subseteq A(X)$; $S(X) \subseteq B(X)$;
- A and B are continuous;
- $\max\{p^2(Sx, Ty), p^2(Ty, Sx)\} \leq \varnothing\{p(Ax, Sx)p(By, Ty), p(Ax, Ty)\, p(By, Sx),$

  $p(Ax, Sx)p(Ax, Ty), p(By, Sx)p(By, Ty),$

  $p(By, Sx)p(Ax, Sx), p(By, Ty)\, p(Ax, Ty)\}$;

  For all $x, y \in X$ and for all p in P, where $\varnothing : [0, \infty)^6 \to (0, +\infty)$ satisfies the following:

- $\varnothing$ is non-decreasing and upper semi-continuous in each coordinate variable and for each $t > 0$

  $\Psi(t)=\max\{\varnothing(t,0,2t,0,0,2t),\varnothing(t,0,0,2t,2t,0),\varnothing(0,t,0,0,0,0),\varnothing(0,0,0,0,0,t)\ \varnothing(0,0,0,0,t,0)\} < t$;

  Then A, B, S and T have a unique common fixed point.

Theorem 2.2: Let A,B,S and T be self maps on a left(right) sequentially complete Quasi-gauge T_0 space (X,P) with condition (iii) and (iv) of Theorem 2.1 such that

- (S,A) ,(A,S),(B,T)and (T,B) are weakly compatible pairs of maps with $T(X) \subseteq A(X)$; $S(X) \subseteq B(X)$;
- One of A,B,S and T is continuous:

  Then the same conclusion of Theorem 2.1 holds.

We prove Theorem 2.1 and Theorem 2.2 without assuming that any function is continuous for finite number of mapping.

3. MAIN RESULTS

Theorem 3.1: Let A, B, S, T, I, J, L, U, P and Q be mappings on left sequentially complete Quasi-gauge T_0 Space (X,P) such that

(P,STJU) and (Q,ABIL) are weakly compatible pairs of mappings with (3.1)

$ABIL(X) \subseteq P(X)$; $STJU(X) \subseteq Q(X)$;

$\max\{p^2(STJUx, ABILy), p^2(ABILy, STJUx)\} \leq \varnothing\{p(Px, STJUx)p(Qy, ABILy),$ (3.2) p(Px, ABILy) p(Qy, STJUx),

p(Px, STJUx)p(Px, ABILy),

p(Qy, STJUx)p(Qy, ABILy),

p(Qy, STJUx)p(Px, STJUx),

p(Qy, ABILy) p(Px, ABILy)};

For all x, y $\in$X and for all p in P, where ø : $[0, \infty)^6 \to (0, +\infty)$ satisfies the following:

ø is non-decreasing and upper semi-continuous in each coordinate variable and for each t >0: (3.3)

$\Psi(t) = \max\{ø(t,0,2t,0,0,2t), ø(t,0,0,2t,2t,0), ø(0,t,0,0,0,0),$

$ø(0,0,0,0,0,t)\ ø(0,0,0,0,t,0)\} < t.$

Then A, B, S, T, I, J, L, U, P and Q have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. since (3.1) holds we can choose x_1, x_2 in X such that Qx_1 = STJUx_0 and Px_2 = ABILx_1.

In general we can choose x_{2n+1} and x_{2n+2} in X such that

$y_{2n} = Qx_{2n+1} = STJUx_{2n}$ and $y_{2n+1} = Px_{2n+2}= ABILx_{2n+1}$; n = 0,1,2 (3.4)

We denote $d_n= p(y_n,y_{n+1})$ and $e_n =p(y_{n+1},y_n)$; now applying (3.2) we get

$\max\{d^2_{2n+2}, e^2_{2n+2}\}$

$= \max\{p^2(STJUx_{2n+2}, ABILx_{2n+3}), p^2(ABILx_{2n+3}, STJUx_{2n+2})\}$

$\leq ø\{p(Px_{2n+2}, STJUx_{2n+2})p(Qx_{2n+3}, ABILx_{2n+3}),$

$p(Px_{2n+2},ABILx_{2n+3})p(Qx_{2n+3},STJUx_{2n+2}),$

$p(Px_{2n+2},STJUx_{2n+2})p(Px_{2n+2},ABILx_{2n+1}),$

$p(Qx_{2n+1},STJUx_{2n+2})p(Qx_{2n+2},ABILx_{2n+3},),$

$p(Qx_{2n+3},STJUx_{2n+2},)\ p(Px_{2n+2},STJUx_{2n+1})$

$p(Qx_{2n+3}, ABILx_{2n+3})\ p(Px_{2n+2}, ABILx_{2n+3})\};$

$=ø\{p(y_{2n+1},y_{2n+2},)p(y_{2n+2},y_{2n+3}),p(y_{2n+1},y_{2n+3})p(y_{2n+2},y_{2n+2}),p(y_{2n+1},y_{2n+2},)p(y_{2n+1},y_{2n+3}),$

$p(y_{2n+2},y_{2n+2})p(y_{2n+2},y_{2n+3},),p(y_{2n+2},y_{2n+2})p(y_{2n+1},y_{2n+2}),p(y_{2n+2},y_{2n+3})p(y_{2n+1},y_{2n+3})\}$

$\leq ø\{ d_{2n+1}, d_{2n+2},0, d_{2n+1}(d_{2n+1}, + d_{2n+2},),0,0, d_{2n+2} (d_{2n+1}+ d_{2n+2})\}.$ (3.5)

If $d_{2n+2} > d_{2n+1}$ then

$\max \{d^2_{2n+1}, e^2_{2n+2}\} \leq ø\{ d^2_{2n+2},0, 2d^2_{2n+2},0,0,2d^2_{2n+2}\} < d^2_{2n+2},$ (3.6)

by (3.3) a contradiction; hence $d_{2n+2}\leq d_{2n+1}$Similarly, we get

$d_{2n+1}\leq d_{2n}.$ (3.7)

By (3.5) and (3.6)

$\max\{d^2_{2n+2}, e^2_{2n+2},\} \leq ø\{ d^2_{2n+1},0, 2d^2_{2n+1},0,0, 2d^2_{2n+1}\}.$

$\leq \Psi(d^2_{2n+1})= \Psi\{P^2(y_{2n+1},y_{2n+2})\}$ (3.8)

Similarly we have

$\max\{d^2_{2n+1}, e^2_{2n+1},\} \leq ø \{ d^2_{2n},0,0, 2d^2_{2n}, 2d^2_{2n},0\}.$

$\leq \Psi \{P^2(y_{2n},y_{2n+1})\}$ (3.9)

So

$d^2_n = P^2(y_n, y_{n+1}) \leq \Psi \{P^2(y_{n+1}, y_n)\} \leq \ldots \leq \Psi^{n-1}\{P^2(y_1, y_2)\}$ (3.10)

and

$e^2_n = P^2(y_{n+1},y_n) \leq \Psi\{P^2(y_{n-1},y_n)\} \leq \ldots \leq \Psi^{n-1}\{P^2(y_1,y_2)\}$ (3.11)

Hence by Lemma 1.1 and from (3.10) and (3.11) we obtain

$\lim d_n = e_n=0.$ (3.12)

Now we prove $\{y_n\}$ is a P-Cauchy sequence. To show $\{y_n\}$ is P-Cauchy it is enough if we show $\{y_{2n}\}$ is P-Cauchy. Suppose $\{y_{2n}\}$ is not a P-Cauchy sequence then there is an $\varepsilon > 0$ such that for each positive integer 2k there exist positive integers 2m(k) and 2n(k) such that for some p in P,

$p(y_{2n(k)},y_{2m(k)}) > \varepsilon$ for $2m(k) >2n(k) >2k$ (3.13)

and

$p(y_{2m(k)},y_{2n(k)}) > \varepsilon$ for $2m(k) >2n(k)>2k$ (3.14)

For each positive even integer 2k, let 2m(k) be the least positive even integer exceeding 2n(k) and satisfying (3.13); hence $p(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon$ then for each even integer 2k,

$\varepsilon < p(y_{2n}(k),y_{2m}(k)$

$\leq p(y_{2m(k)},y_{2n(k)-2})+ (d_{2m(k)-2} + d_{2m(k)-1})$ (3.15)

From (3.12) and (3.15), we obtain $\lim p(y_{2n(k)},y_{2m(k)}) = \varepsilon$. By the triangle inequality

$p(y_{2n(k)},y_{2m(k)}) \leq p(y_{2n(K)}, y_{2m(k)-1})+ d_{2m(k)-1}\}$

$p(y_{2n(K)}, y_{2m(k)-1}) \leq p(y_{2n(k)},y_{2m(k)})+ e_{2m(k)-1};$

So

$p(y_{2n(k)},y_{2m(k)})-p(y_{2n(k)},y_{2m(k)-1}) | \leq \max\{ d_{2m(k)-1}, e_{2m(k)-1}\}.$ (3.16)

Similarly By triangle inequality

$p(y_{2n(K)+1},y_{2m(k)-1}) - p(y_{2n(K)},y_{2m(k)}) | \leq \max\{ e_{2n(k)}+ e_{2m(k)-1}, d_{2n(k)}+ d_{2m(k)-1}\}.$ (3.17)

From (3.16) and (3.17) as $k\to\infty$,$\{p(y_{2n(K)},y_{2m(k)-1})\}$ and $p(y_{2n(K)+1},y_{2m(k)-1})\}$ converge to ε.

Similarly if $p(y_{2m(K)}, y_{2n(k)}) > \varepsilon$,

$\lim p(y_{2m(K)}, y_{2n(k)}) = \lim p(y_{2m(K)-1}, y_{2n(k)+1})$

$= \lim p(y_{2m(K)-1}, y_{2n(k)}) = \varepsilon$ as $k \to \infty$.

By (3.2)

$\varepsilon < p(y_{2n(K)}, y_{2m(k)})$

$\leq p(y_{2n(K)}, y_{2n(k)+1}) + p(y_{2n(K)+1}, y_{2m(k)})$

$\leq d_{2n(k)} + \max\{ p(y_{2n(K)+1}, y_{2m(k)}), p(y_{2n(K)}, y_{2n(k)+1})\}$

$= d_{2n(k)} + \max\{p(ABIL\ x_{2n(k)+1}, STJU\ x_{2m(k)}), p(STJU\ x_{2m(k)}, ABIL\ x_{2n(k)+1})\}$

$\leq d_{2n(k)} + [\varnothing\{p(y_{2m(K)-1}, y_{2m(k)}) p(y_{2n(K)}, y_{2n(k)+1}),$

$p(y_{2m(K)-1}, y_{2n(k)+1})\ p(y_{2n(K)}, y_{2m(k)}),$

$p(y_{2m(K)-1}, y_{2m(k)})\ p(y_{2m(K)-1}, y_{2n(k)+1}),$

$p(y_{2n(K)}, y_{2m(k)})\ p(y_{2n(K)}, y_{2n(k)+1}),$

$p(y_{2n(K)}, y_{2m(k)})\ p(y_{2m(K)-1}, y_{2m(k)}),$

$p(y_{2n(K)}, y_{2n(k)+1})\ p(y_{2m(K)-1}, y_{2n(k)+1})\}]^{1/2}$

Since $\varnothing$ is upper semi-continuous, as $k \to \infty$ we get that $\varepsilon \leq \{\varnothing(0, \varepsilon^2, 0, 0, 0, 0)\}^{1/2} < \varepsilon$, which is a contradiction. Therefore $\{y_n\}$ is P-Cauchy sequence in X. Since X is complete there exists a point z in X such that $\lim n \to \infty\ y_n = z$.

$\lim n \to \infty\ Px_{2n} = \lim n \to \infty\ ABILx_{2n-1} = z$

and

$\lim n \to \infty\ Qx_{2n+1} = \lim n \to \infty\ STJUx_{2n-2} = z$

Since $STJU(X) \subseteq Q(X)$, there exist a point $u \in X$ such that $z = Qu$. Then using (3.2),

$\max\{p^2(STJUx_{2n}, ABILu), p^2(ABILu, STJUx_{2n})\}$

$\leq \varnothing\{p(Px_{2n}, STJUx_{2n}) p(Qu, ABILu), p(Px_{2n}, ABILu) p(Qu, STJUx_{2n}),$

$p(Px_{2n}, STJUx_{2n})\ p(Px_{2n}, ABILu), p(Qu, STJUx_{2n})\ p(Qu, ABILu),$

$p(Qu, STJUx_{2n})\ p(Px_{2n}, STJUx_{2n}), p(Qu, ABILu)\ p(Px_{2n}, ABILu)\ \};$

Taking limit as $n \to \infty$,

$\max\{p^2(z, ABILu), p^2(ABILu, z)\}$

$\leq \varnothing\{p(z, z) p(z, ABILu), p(z, ABILu) p(z, z), p(z, z) p(z, ABILu),$

$p(z, z)\ p(z, ABILu), p(z, z) p(z, z), p(z, ABILu)\ p(z, ABILu)\}$

$\leq \varnothing\{0, 0,0,0,0, p(z, ABILu)\ p(z, ABILu)\},$

$< p(z, ABILu)\ p(z, ABILu)$

a contradiction. Thus ABILu = z. Therefore ABILu = z = Qu.

Similarly, since ABIL(X) ⊆ P(X), there exist a point $\upsilon \in X$, such that $z = P\upsilon$.

Then using (3.2),

$\max\{p^2(STJU\upsilon, ABILx_{2n+1}), p^2(ABILx_{2n+1},STJU\upsilon)\}$

$\leq ø\{p(P\upsilon,STJU\upsilon)p(Qx_{2n+1},ABILx_{2n+1}),p(P\upsilon,ABILx_{2n+1})p(Qx_{2n+1},STJU\upsilon),$

$p(P\upsilon,STJU\upsilon)\, p(P\upsilon,ABILx_{2n+1}), p(Qx_{2n+1},STJU\upsilon)\, p(Qx_{2n+1},ABILx_{2n+1}),$

$p(Qx_{2n+1},STJU\upsilon)\, p(P\upsilon,STJU\upsilon), p(Qx_{2n+1},ABILx_{2n+1})\, p(P\upsilon,ABILx_{2n+1})\}$

Taking limit as $n\to\infty$,

$\max\{p^2(STJU\upsilon, z), p^2(z,STJU\upsilon)\}$

$\leq ø\ \{p(z,STJU\upsilon)\, p(z,z), p(z,z)\, p(z,STJU\upsilon),$

$p(z,STJU\upsilon)\, p(z,z),p(z,STJU\upsilon)\, p(z,z),$

$p(z,STJU\upsilon)\, p(z,STJU\upsilon),p(z,z)\, p(z,z)\}$

$\leq ø\{0,0,0,0,p(z, STJU\upsilon)\, p(z,STJU\upsilon),0\}$

$< p(z,STJU\upsilon,p(z,STJU\upsilon)$

a contradiction. Thus $z = STJU\upsilon$. Therefore $z = STJU\upsilon = P\upsilon$.

Hence, $z = Qu = ABILu = P\upsilon = STJU\upsilon$.

Since the pair of mappings Q and ABIL are Weakly Compatible, then QABILu = ABILQu.

i.e. Qz = ABILz. Now we show that z is a fixed point of ABIL.

If $ABILz \neq z$, then by (3.2)

$\max\{p^2(STJUx_{2n} , ABILz), p^2(ABILz,STJUx_{2n})\}$

$\leq ø\{p(Px_{2n},STJUx_{2n})p(Qz,ABILz),p(Px_{2n},ABILz)p(Qz,STJUx_{2n}),$

$p(Px_{2n},STJUx_{2n})p(Px_{2n},ABILz),p(Qz,STJUx_{2n})p(Qz,ABILz),$

$p(Qz,STJUx_{2n})\, p(Px_{2n},STJUx_{2n}),p(Qz,ABILz)\, p(Px_{2n},ABILz)\ \};$

Taking limit as $n\to\infty$,

$\max\{p^2(z, ABILz), p^2(ABILz,z)\}$

$\leq ø\{p(z,z)p(Qz,ABILz),p(z,ABILz)p(Qz,z),$

$p(z,z)\, p(z,ABILz),p(Qz,z)p(Qz,ABILz),$

$p(Qz,z)\, p(z,z), p(Qz,ABILz)p(z,ABILz)\};$

$\leq ø\{0, 0,0,0,0, p(z,ABILz)\, p(z,ABILz)\},$

< p(z,ABILz) p(z,ABILz)

a contradiction. Thus ABILz = z. Therefore ABILz = z = Qz.

Similarly we prove that STJUz = z = Pz.

Hence Pz = Qz = STJUz = ABILz = z; thus z is a common fixed point of A,B,S,T,I,J,L,U,P and Q. Uniqueness follows trivially. Therefore z is a unique common fixed point of A, B, S, T, I, J, L, U, P and Q.

ACKNOWLEDGEMENTS

The authors are thankful to the Principal and Additional Director, Dr. Usha Shrivastav for her kind cooperation.

REFERENCES

1. Antony, J., Studies in fixed points and Quasi-Gauges, Ph.D. Thesis, I.I.T., Madras (1991).
2. Antony, Jessy and Subrahmanyam, P.V., Quasi-Gauges and Fixed Points, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 24 (1), (1994), 31-42.
3. Fisher, B., Theorems on common fixed points, Fund. Math. 113 (1981), 37-43.
4. Matkowski, J., Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 62 (1977), No.2, 344-348.
5. Pathak, H. K., Chang, S. S. and Cho, Y. J., Fixed point theorems for compatible mappings of type (P), Indian J. Math. 36 (1994), No. 2, 151-166.
6. Rao, I. H. N. and Murty, A. S. R., Common fixed point of weakly compatible mappings in Quasi-gauge spaces, J. Indian Acad. Math. 21(1999), No. 1, 73-87.
7. Reilly, I.L.., Quasi-gauge spaces. J. Lond. Math. Soc. 6 (1973), 481-487.